(Theorem):- Prove that a monotone increasing sequence tends to its upper bound.

Proof:- Let $\{a_n\}$ be a monotone increasing sequence whose upper bound is M. Here, two cases arise:—

(i) M is finite

(ii) M is infinite.

When M is finite [By def^n of least upper bound]

(i) $a_n \leq M$ for all n. —— (1)

(ii) $a_n + \epsilon > M$ for ~~all~~ at least one n. Let us be true for $n = m$.

Then, $a_m + \epsilon > M$

i.e. $a_m > M - \epsilon$ —— (2)

But the sequence is monotone increasing.

So, $a_n \geq a_m$ if $n \geq m$ —— (3)

$\therefore$ From (2) & (3), we have

$a_n > M - \epsilon$ for all $n \geq m$.

Again, from (1), $a_n \leq M$ for all n.

So, $a_n < M + \epsilon$ for all n and hence also for $n \geq m$.

$\therefore$ Combining these, $M - \epsilon < a_n < M + \epsilon$ for $n \geq m$.

$\therefore |a_n - M| < \epsilon$ for $n \geq m$.

$\therefore \{a_n\}$ tends to M.

When M is infinite, then from (2),

$a_m > M - \epsilon = N$ (when N is arbitarily large).

But $a_n \geq a_m > N$ for $n \geq m$.

$\therefore \{a_n\}$ tends to ∞ and so to the upper bound ∞.

(Theorem):- Prove that a monotone decreasing sequence tends to its lower bound.

Proof:- Let $\{a_n\}$ be a monotone decreasing sequence whose greatest lower bound is K. Here, two cases arise

(i) K is finite,

(ii) K is infinite.

When K is finite [By def^n of greatest lower bound, we have,

(i) $a_n \geq K$ for all n, —— ①

(ii) $a_n < K + \epsilon$ for at least n.

Let it be true for $n = m$. Then,

$$a_m < K + \epsilon \text{ —— ②}$$

But $\{a_n\}$ is monotone decreasing sequence.

So, $a_n \leq a_m$ for all $n \geq m$ —— ③

From ② & ③, we have

$a_n < K + \epsilon$ for $n \geq m$.

and from ①, $a_n > K - \epsilon$ for all n and so for $n \geq m$.

Combining these, $K - \epsilon < a_n < K + \epsilon$ for $n \geq m$.

$\therefore |a_n - K| < \epsilon$, for $n \geq m$.

Hence, $\{a_n\}$ tends to the greatest lower bound K.

When K is infinite, If the lower bound is infinite, it must be negative & given any +ve no. G ~~we~~ we have $a_n < -G$ for at least one value of n, let it be m.

So, $a_m < -G$, But $a_n \leq a_m$, for $n \geq m$.

$\therefore a_n < -G$, for $n \geq m$.

Hence, $a_n \to \infty$ as $n \to \infty$.

Q.No.- Give an example of a monotonic increasing Sequence which is Convergent.

Ans.- We Know that the sequence is monotonic increasing when $u_n < u_{n+1}$

let us Consider a Sequence $\{u_n\}$ where $u_n = \frac{n}{n+1}$

Here, $u_1 = \frac{1}{2}$, $u_2 = \frac{2}{3}$, $u_3 = \frac{3}{4}$.

we find that,

$$u_1 < u_2 < u_3 - - - -$$

Hence, sequence $\{u_n\}$ is monotonic increasing

Now, we have to Shown that $\{u_n\}$ is Convergent.

$$\therefore u_n = \frac{n}{n+1}$$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{1+\frac{1}{\infty}} = 1 \text{ which is finite.}$$

Hence, given Sequence $\{u_n\}$ is Convergent

Q.No.- Give an example of monotonic increasing sequence which is divergent.

Ans.- We Know that a Sequence $\{u_n\}$ is monotonic increasing when $u_n < u_{n+1}$

i.e. $u_1 < u_2 < u_3 < - - - -$

The sequence $\{u_n\}$ when $u_n = n$ is monotonic increasing. As.

$\because u_1 = 1$

$u_2 = 2$

$u_3 = 3$

Here, $u_1 < u_2 < u_3 - - - -$

$\because u_n = n$

$= \lim_{n\to\infty} u_n = \lim_{n\to\infty} n$

$= \infty$

Hence sequence $\{n\}$ diverges to ∞.

QNo- Discuss the converges of the sequence $\{a_n\}$ defind by

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + - - - + \frac{1}{n-n}$$

Ans. $\because$ It is given that.

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + - - - - + \frac{1}{n+n-1} + \frac{1}{n+n} \quad -(1)$$

$$\therefore a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + - - - \frac{1}{(n+1)+n} + \frac{1}{(n+1)+(n-1)} \quad -(2)$$

(2) − (1)

$$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{2(n+1) + 2n+1 - 2(2n+1)}{2(2n+1)(n+1)}$$

$$= \frac{\cancel{2n} + \cancel{2} + \cancel{2n} + 1 - \cancel{4n} - \cancel{2}}{2(2n+1)(n+1)}$$

$$a_{n+1} - a_n = \frac{1}{2(2n+1)(n+1)} > 0 \quad \text{for all } n.$$

$$\therefore a_{n+1} > a_n$$

Hence the Sequence $\{a_n\}$ is monotonic increasing.

$$\therefore a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

$$a_n < \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$$

$$a_n < \cancel{n}\frac{1}{\cancel{n}}$$

$$a_n < 1.$$

Hence, the Sequence $\{a_n\}$ is bounded above and monotonic increasing.

We known that monotonic increasing Sequence is Convergent.

Hence the given Sequence is Convergent.